Existence of minimizers for Schrödinger operators under domain perturbations with application to Hardy's inequality

Yehuda Pinchover

Department of Mathematics

Technion - Israel Institute of Technology

Haifa 32000, Israel

pincho@techunix.technion.ac.il

Kyril Tintarev

Department of Mathematics

Uppsala University

SE-751 06 Uppsala, Sweden

kyril.tintarev@math.uu.se

February 1, 2008

Abstract

The paper studies the existence of minimizers for Rayleigh quotients $\mu_\Omega = \inf \frac{\int_\Omega |\nabla u|^2}{\int_\Omega V|u|^2}$, where Ω is a domain in $\mathbb{R}^N$, and V is a nonzero nonnegative function that may have singularities on $\partial\Omega$. As a model for our results one can take Ω to be a Lipschitz cone and V to be the Hardy potential $V(x) = \frac{1}{|x|^2}$.

2000 *Mathematics Subject Classification.* Primary 35J70; Secondary 35J20, 49R50.

Keywords. concentration compactness, gap phenomenon, Hardy inequality, principal eigenvalue.

1 Introduction

Let X be a domain in $\mathbb{R}^N$, and let $V \in L^p_{loc}(X)$ be a nonzero nonnegative function, where $p > \frac{N}{2}$. Let $\mathcal{D}^{1,2}(X)$ be the completion of $C_0^\infty(X)$ with respect to the norm $\|u\|^2 = \int_X |\nabla u|^2$. For an open set $\Omega \subset X$, we will consider the subspace $\mathcal{D}^{1,2}(\Omega) \subset \mathcal{D}^{1,2}(X)$, which is by definition, the closure in $\mathcal{D}^{1,2}(X)$ of $C_0^\infty(\Omega)$. We denote $B \Subset X$, if $B \subset X$, and $\overline{B}$ is compact in X.

Let $\Omega \subset X$. We study the existence of a minimizer for the Rayleigh quotient

$$\mu_\Omega = \inf_{u \in \mathcal{D}^{1,2}(\Omega),\, Vu \not\equiv 0} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega V|u|^2}, \qquad (1.1)$$

under the assumption that

$$\mu_X > 0. \qquad (1.2)$$

Condition (1.2) is satisfied, for example, when $X = \mathbb{R}^N \setminus \{0\}$, $V(x) = \frac{1}{|x|^2}$, and $N \geq 3$, which corresponds to the well-known Hardy inequality, with $\mu_X = \frac{(N-2)^2}{4}$. Existence of a minimizer in problems with a singular potential has been studied by many authors with attention to 'small' perturbations of the potential V (see, [2, 3, 4, 6, 7, 13, 14] and the references therein). Typically in such cases, if there is a 'spectral gap', then a minimizer exists. This situation is sometimes called the 'gap phenomenon'. The present paper studies the existence of a minimizer in the case of compact domain perturbations under the situation of a positive 'spectral gap'. Domain perturbations in the context of variational inequalities and the Dirichlet problem were studied in [5, 8] and the references therein.

Let P be a second order elliptic operator which is defined on a domain Ω, and denote by $\mathcal{C}_P(\Omega)$ the cone of all positive solutions of the equation $Pu = 0$ in Ω. For $P_\mu := -\Delta - \mu V$, we simply write $\mathcal{C}_\mu(\Omega) := \mathcal{C}_{P_\mu}(\Omega)$. Let $K \Subset \Omega$. Recall [11, 12] that $u \in \mathcal{C}_P(\Omega \setminus K)$ is said to be a *positive solution of the operator P of minimal growth in a neighborhood of infinity in Ω*, if for any $K \Subset K_1 \Subset \Omega$ and any $v \in C(\overline{\Omega \setminus K_1}) \cap \mathcal{C}_P(\Omega \setminus K_1)$, the inequality $u \leq v$ on ∂K_1 implies that $u \leq v$ in $\Omega \setminus K_1$. A positive solution $u \in \mathcal{C}_P(\Omega)$ which has minimal growth in a neighborhood of infinity in Ω is called a *ground state* of P in Ω.

The operator P is said to be *critical* in Ω, if P admits a ground state in Ω. The operator P is called *subcritical* in Ω, if $\mathcal{C}_P(\Omega) \neq \emptyset$, but P is not critical in Ω. If $\mathcal{C}_P(\Omega) = \emptyset$, then P is *supercritical* in Ω.

Suppose that P is critical in $\Omega \subsetneq X$. Then P is subcritical in any domain Ω_1 such that $\Omega_1 \subsetneq \Omega$, and supercritical in any domain Ω_2 such that $\Omega \subsetneq \Omega_2 \subset X$. Furthermore, for any nonzero nonnegative function W the operator $P + W$ is subcritical and $P - W$ is supercritical in Ω. Moreover, if P is critical in Ω, then $\dim \mathcal{C}_P(\Omega) = 1$ (see e.g. [12]).

If P is subcritical in Ω, then P admits a positive minimal Green function $G_P^\Omega(x, y)$ in Ω. Moreover, for each $y \in \Omega$, the function $G_P^\Omega(\cdot, y)$ is a positive

solution of the equation $Pu = 0$ in $\Omega \setminus \{y\}$ that has minimal growth in a neighborhood of infinity in Ω (see [12]).

Consider now the case that $P_\mu = -\Delta - \mu V$, where V is a nonzero nonnegative function and $\mu \in \mathbb{R}$. Then P_μ is subcritical in Ω for all $\mu < \mu_\Omega$, supercritical in Ω for all $\mu > \mu_\Omega$, and P_{μ_Ω} is either critical or subcritical, where μ_Ω is defined by (1.1).

In many papers the term *ground state* refers only to minimizer solutions of (1.1). It turns out that such a minimizer solution is also the ground state of the operator $-\Delta - \mu_\Omega V$ in the sense introduced above. For Schrödinger operators this fact was proved in [9] (see Theorem 2.7 therein, and the remark below its proof). The following lemma applies also to the general symmetric case, and its proof applied even to nonsymmetric cases. An alternative proof that was suggested to us by M. Murata (after the first draft of the present paper has been completed) uses the heat kernel.

Lemma 1.1. *Suppose that $V > 0$ and (1.1) admits a minimizer, then the operator $-\Delta - \mu_\Omega V$ is critical in Ω, and a minimizer is a ground state.*

Our first main result reads as follows.

Theorem 1.2. *Suppose that $\Omega \subset X$ is a domain satisfying $0 < \mu_X < \mu_\Omega$. Then there exists an open set $B \Subset X$ such that $\Omega \cup B$ is connected and*

$$\mu_{\Omega \cup B} < \mu_\Omega. \tag{1.3}$$

Moreover, for any such set B the infimum value $\mu_{\Omega \cup B}$ for problem (1.1) is uniquely attained.

Corollary 1.3. *Suppose that B satisfies the conditions of Theorem 1.2. Then for every open set B' such that $B \subset B' \Subset X$ and $\Omega \cup B'$ is connected, the infimum value $\mu_{\Omega \cup B'}$ is attained.*

Proof. The inclusion $\mathcal{D}^{1,2}(\Omega \cup B) \subset \mathcal{D}^{1,2}(\Omega \cup B')$ implies $\mu_{\Omega \cup B'} \leq \mu_{\Omega \cup B} < \mu_\Omega$, and hence Theorem 1.2 applies. $\square$

In the critical case we have the following stronger statement.

Theorem 1.4. *Suppose that $\Omega \subset X$ is a domain satisfying $0 < \mu_X < \mu_\Omega$, and assume that the operator $P = -\Delta - \mu_\Omega V$ is critical in Ω.*

Then for any open set $B \Subset X$ such that $\Omega \cup B$ is connected and $\Omega \neq \Omega \cup B$, the inequality (1.3) is satisfied, and the infimum value $\mu_{\Omega \cup B}$ for problem (1.1) is uniquely attained.

3

If $\Omega \Subset X$, then it is well known that μ_Ω in (1.1) is attained since $\int_\Omega V|u|^2$ is weakly continuous. For a noncompact domain Ω, or a potential V that blows up near $\partial\Omega \cap \partial X$ the minimizer may not exist, as the following example demonstrates.

Example 1.5. Consider a Lipschitz (connected) cone $C \subset \mathbb{R}^N \setminus \{0\}$, $N \geq 2$, with the vertex at 0. Let $V(x) = \frac{1}{|x|^2}$, and $\mu \in \mathbb{R}$. Denote by $\mathcal{C}_\mu^0(C)$ the cone of all positive solutions of the equation

$$P_\mu u := -\Delta u - \mu \frac{u}{|x|^2} = 0 \tag{1.4}$$

in C that vanish on $\partial C \setminus \{0\}$. By [11], the dimension of $\mathcal{C}_\mu^0(C)$ is at most 2. Actually, using separation of variables and [11], one can compute the solutions in $\mathcal{C}_\mu^0(C)$ explicitly.

Let $D \subset S_1^{N-1}$ be the Lipschitz domain so that

$$C = \{(r, \omega) \mid r \in (0, \infty), \omega \in D\}.$$

Denote by Δ_r and Δ_S the radial and the spherical Laplacian, respectively. Let λ_D and $v_D(\omega)$ be the Dirichlet principal eigenvalue and eigenfunction of $-\Delta_S$ on D. So,

$$-\Delta_S v_D = \lambda_D v_D \quad \text{on } D, \quad v_D|_{\partial D} = 0.$$

Then any positive solution in $\mathcal{C}_\mu^0(C)$ is of the form

$$u_{\mu,D}(r)v_D(\omega) \quad r \in (0, \infty), \omega \in S_1^{N-1},$$

where $u_{\mu,D}$ is a *global* positive solution of the Euler equidimensional equation

$$-\Delta_r u - \frac{\mu - \lambda_D}{r^2} u = -u'' - \frac{(N-1)}{r} u' - \frac{\mu - \lambda_D}{r^2} u = 0 \quad 0 < r < \infty.$$

It follows that μ should satisfy $\mu \leq \frac{(N-2)^2}{4} + \lambda_D$, and $u_{\mu,D}(r) = ar^{\alpha_+} + br^{\alpha_-}$, where

$$\alpha_\pm = \alpha_\pm(\mu, D) = \frac{-(N-2) \pm \sqrt{(N-2)^2 - 4(\mu - \lambda_D)}}{2},$$

and $a, b \geq 0$. In particular,

$$\mu_C = \frac{(N-2)^2}{4} + \lambda_D, \tag{1.5}$$

4

and the corresponding unique positive solution in $\mathcal{C}^0_{\mu_C}(C)$ equals $r^{-\frac{(N-2)}{2}}v_D(\omega)$, which clearly does not belong to $\mathcal{D}^{1,2}(C)$. It is well known that if a minimizer of the variational problem exists, then it belongs to $\mathcal{C}^0_{\mu_C}(C)$. Therefore, μ_C is not attained for any Lipschitz cone $C \subset \mathbb{R}^N$. On the other hand, noting that the solution $r^{-\frac{(N-2)}{2}}\log r\, v_D(\omega)$ is a positive solution of the equation $P_{\mu_C}u = 0$ near $\zeta = 0$ and $\zeta = \infty$ which grows there faster than $r^{-\frac{(N-2)}{2}}v_D(\omega)$, and using [11], it follows that $r^{-\frac{(N-2)}{2}}v_D(\omega)$ is a ground state of the *critical* operator P_{μ_C} in C.

Now, for $N \geq 3$ take $X := \mathbb{R}^N \setminus \{0\}$, and note that $\mu_X = \frac{(N-2)^2}{4} > 0$, so, (1.2) is satisfied. For $N = 2$ take a Lipschitz cone X with a vertex at the origin such that $\overline{C} \setminus \{0\} \subset X \subsetneq \mathbb{R}^N \setminus \{0\}$. So, (1.2) is satisfied also in the two dimensional case.

Consequently, Theorem 1.4 implies that for any open set $B \Subset X$ such that $C \cup B$ is connected, and $C \subsetneq C \cup B$, the infimum value $\mu_{C \cup B}$ is uniquely attained. By [11], it follows that the corresponding minimizer behaves near $\zeta = \infty$ and near $\zeta = 0$ like $r^{\alpha_-(\mu_{C \cup B},D)}v_D(\omega)$ and $r^{\alpha_+(\mu_{C \cup B},D)}v_D(\omega)$, respectively.

On the other hand, if B is replaced by a larger set that is not relatively compact in X, then a minimizer may not exist. Take for example two connected Lipschitz cones C and C_1, such that $C \subsetneq C_1 \subset X$. Notice that one has $\lambda_{D_1} < \lambda_D$, and by (1.5), $\mu_{C_1} < \mu_C$. Hence, for $B = C_1$ we have $C \cup B = C_1$, and consequently, the infimum $\mu_{C \cup B}$ is not attained.

Next, we discuss the subcritical case, where adding a compact set that is too small, also implies the non-existence of a minimizer:

Theorem 1.6. *Let $\Omega \subsetneq X$ be a domain with a Lipschitz boundary, and let $V \in C^\alpha_{loc}(X)$ be a positive function, where $0 < \alpha \leq 1$. Assume that the operator $P := -\Delta - \mu_\Omega V$ is subcritical in Ω, and (1.2) is satisfied.*

Let $B_j \Subset X$ be a decreasing sequence of smooth domains, such that $\Omega_j := B_j \cup \Omega$ are connected for all $j \geq 1$, $\mathrm{int}\,(\cap_j \Omega_j) = \Omega$, and $B_1 \cap \partial\Omega$ is contained in a Lipschitz portion $\Gamma \Subset \partial\Omega$. Then there exists $j_0 > 0$ such that for all $j \geq j_0$, μ_{Ω_j} is not attained. Moreover, $-\Delta - \mu_\Omega V$ is subcritical in Ω_j for all $j \geq j_0$.

In particular, we have

Corollary 1.7. *Let $C \subsetneq X$ be a Lipschitz cone with vertex at 0, where $X = \mathbb{R}^N \setminus \{0\}$ if $N \geq 3$, and $X \subsetneq \mathbb{R}^N \setminus \{0\}$ is a Lipschitz cone with a*

vertex at the origin such that $\overline{C} \setminus \{0\} \subset X$, *if* $N = 2$. *Let* $W \in C_{loc}^{\alpha}(X)$, $0 < \alpha \leq 1$ *be a nonzero nonnegative function with a compact support in* C, *and set* $V(x) = \frac{1}{|x|^2} - W(x)$. *Let* $B_j \Subset X$ *be a decreasing sequence of smooth domains, such that* $C_j := B_j \cup C$ *are connected for all* $j \geq 1$, *and* int $(\cap_j C_j) = C$. *Then there exists* $j_0 > 0$ *such that for all* $j \geq j_0$, μ_{C_j} *is not attained, and* $-\Delta - \mu_C V$ *is subcritical in* C_j *for all* $j \geq j_0$.

2 Existence of minimizers under compact domain perturbations

In this section we prove Theorem 1.2 and Theorem 1.4. Throughout the section we assume that $\mu_X < \mu_\Omega$.

Lemma 2.1. *For any* $\varepsilon > 0$ *there exists an open bounded set* $B_\varepsilon \Subset X$, *such that* $\mu_{B_\varepsilon} \leq \mu_X + \varepsilon$.

Proof. Since $C_0^\infty(X)$ is dense in $\mathcal{D}^{1,2}(X)$, there exists a minimizing sequence $u_k \in C_0^\infty(X)$, such that $\int_X V|u_k|^2 = 1$ and $\|u_k\|^2 \leq \mu_X + k^{-1}$. Fix $k_\varepsilon > \varepsilon^{-1}$, and choose an open bounded set B_ε so that supp $u_{k_\varepsilon} \subset B_\varepsilon \Subset X$. Then $u_{k_\varepsilon} \in \mathcal{D}^{1,2}(B_\varepsilon)$ and $\mu_{B_\varepsilon} \leq \|u_{k_\varepsilon}\|^2 \leq \mu_X + k_\varepsilon^{-1} < \mu_X + \varepsilon$. $\qquad\square$

Let $0 < \varepsilon < \mu_\Omega - \mu_X$. Since $\mathcal{D}^{1,2}(B_\varepsilon) \subset \mathcal{D}^{1,2}(\Omega \cup B_\varepsilon)$, we have

$$\mu_{\Omega \cup B_\varepsilon} \leq \mu_{B_\varepsilon} < \mu_X + \varepsilon < \mu_\Omega.$$

Recall that if the operator $P = -\Delta - \mu_\Omega V$ is critical in Ω, and $\Omega \subsetneqq \Omega_1$, then $\mu_{\Omega_1} < \mu_\Omega$. Consequently, the assertions of theorems 1.2 and 1.4 follow from the following statement.

Lemma 2.2. *If* $B \Subset X$ *is an open set, and* $\mu_{\Omega \cup B} < \mu_\Omega$, *then* $\mu_{\Omega \cup B}$ *is attained and every minimizing sequence for* $\mu_{\Omega \cup B}$ *is convergent.*

Proof. Let $\{u_k\}$ be a minimizing sequence for $\mu_{\Omega \cup B}$. So, we may assume that $\int_{\Omega \cup B} V|u_k|^2 = 1$ and $\|u_k\|^2 \to \mu_{\Omega \cup B}$. Consider a weakly convergent in $\mathcal{D}^{1,2}(\Omega \cup B)$ subsequence of $\{u_k\}$, which we relabel as $\{u_k\}$. Let $w :=$ w-lim u_k, and denote $v_k := u_k - w \rightharpoonup 0$. Since $(v_k, w) \to 0$, we have

$$\|u_k\|^2 = \|v_k + w\|^2 = \|v_k\|^2 + \|w\|^2 + 2\text{Re}\,(v_k, w) = \|v_k\|^2 + \|w\|^2 + o(1), \quad (2.1)$$

6

so that
$$\|v_k\|^2 + \|w\|^2 = \mu_{\Omega \cup B} + o(1). \tag{2.2}$$

Note that $\int_{\Omega \cup B} V v_k w \to 0$, since (1.2) and Cauchy-Schwartz inequality imply that $u \mapsto \int_{\Omega \cup B} V u w$ is a continuous functional on $\mathcal{D}^{1,2}(\Omega \cup B)$. Thus, by repeating the derivation of (2.2) for the seminorm $\sqrt{\int V|u|^2}$, we have,

$$\int_{\Omega \cup B} V|v_k|^2 + \int_{\Omega \cup B} V|w|^2 = 1 + o(1). \tag{2.3}$$

Let $t = \int_{\Omega \cup B} V w^2$. Once we show that

$$\|v_k\|^2 \geq \mu_\Omega \int_{\Omega \cup B} V|v_k|^2 + o(1), \tag{2.4}$$

we will have from (2.2) and (2.3) that $(1 - t)\mu_\Omega + t\mu_{\Omega \cup B} \leq \mu_{\Omega \cup B}$. Since $\mu_\Omega > \mu_{\Omega \cup B}$, this can hold only if $t = 1$. By (2.2), $\mu_{\Omega \cup B} \geq \|w\|^2$ and since $\int_{\Omega \cup B} V|w|^2 = 1$ we see that w is a minimizer. Moreover, since $\|u_k\| \to \|w\|$, $u_k \to w$ in $\mathcal{D}^{1,2}$.

Let us verify (2.4). Let $\chi \in C_0^\infty(X; [0,1])$ be equal 1 on $\overline{B}$. Then, by the compactness of the Sobolev imbedding on bounded smooth sets, we have $\int_{\mathrm{supp}\,\chi} V|v_k|^2 \to 0$, and

$$\int_{\Omega \cup B} V|v_k|^2 = \int_{\Omega \cup B} V[(1-\chi)^2|v_k|^2 + \chi(2-\chi)|v_k|^2] = \int_\Omega V|(1-\chi)v_k|^2 + o(1). \tag{2.5}$$

Observe that

$$\int_{\Omega \cup B} |\nabla v_k|^2 - \int_{\Omega \cup B} |\nabla((1-\chi)v_k)|^2 =$$

$$- \int_{\Omega \cup B} \left(|\nabla\chi|^2|v_k|^2 - 2(1-\chi)v_k\nabla\chi \cdot \nabla v_k\right) +$$

$$\int_{\Omega \cup B} \chi(2-\chi)|\nabla v_k|^2. \tag{2.6}$$

By the compactness of Sobolev imbedding on relatively compact smooth sets, we have

$$\int_{\Omega \cup B} |\nabla\chi|^2|v_k|^2 \leq C \int_{\mathrm{supp}\,\chi} |v_k|^2 = o(1), \tag{2.7}$$

and

$$\left| \int_{\Omega \cup B} (1-\chi) v_k \nabla \chi \cdot \nabla v_k \right| \leq C \left(\int_{\Omega \cup B} |\nabla v_k|^2 \right)^{\frac{1}{2}} \left(\int_{\operatorname{supp} \chi} |v_k|^2 \right)^{\frac{1}{2}} = o(1).$$
(2.8)

Combining (2.6),(2.7) and (2.8), we have

$$\|v_k\|^2 \geq \int_{\Omega \cup B} |\nabla ((1-\chi) v_k)|^2 + o(1).$$
(2.9)

Claim: For any $\psi \in \mathcal{D}^{1,2}(\Omega \cup B)$, we have $(1-\chi)\psi \in \mathcal{D}^{1,2}(\Omega)$. Let $\{\psi_l\}_{l=1}^{\infty} \subset C_0^{\infty}(\Omega \cup B)$ be a sequence such that $\psi_l \to \psi$ in $\mathcal{D}^{1,2}(\Omega \cup B)$. Since $(1-\chi)\psi_l \in \mathcal{D}^{1,2}(\Omega)$, it is enough to show that $(1-\chi)\psi_l \to (1-\chi)\psi$ in $\mathcal{D}^{1,2}(\Omega)$.

Indeed

$$\int_{\Omega} |\nabla ((1-\chi)(\psi_l - \psi))|^2 \leq$$

$$2 \int_{\Omega} |\nabla(1-\chi)|^2 |\psi_l - \psi|^2 + 2 \int_{\Omega} |(1-\chi)|^2 |\nabla(\psi_l - \psi)|^2 \leq$$

$$2 \int_{\Omega} |\nabla(1-\chi)|^2 |\psi_l - \psi|^2 + 2 \int_{\Omega \cup B} \nabla(\psi_l - \psi)|^2 \to 0,$$

where we used the compactness of Sobolev imbedding on relatively compact smooth sets.

By the Claim $(1-\chi)v_k \in \mathcal{D}^{1,2}(\Omega)$, therefore, (2.9) and the definition of μ_Ω imply

$$\|v_k\|^2 \geq \mu_\Omega \int_{\Omega} V |(1-\chi) v_k|^2 + o(1).$$
(2.10)

Substituting (2.5) into the last inequality, we obtain (2.4), which proves the lemma. $\qquad \square$

3 Proof of Lemma 1.1

Throughout this section, Ω denotes a domain in $\mathbb{R}^N$, $N \geq 2$, and $V > 0$. We start with a brief discussion of some spectral properties of the operator $P = -V(x)\Delta$ in Ω.

First, we turn Ω into a Riemannian manifold M equipped with the metric $ds^2 = (V(x))^{-1} \sum_{i=1}^{N} dx_i^2$. We put $\tilde{L}_2(M) = L_2(\Omega; V)$ equipped with the

norm $|u|_2 = (\int_\Omega |u|^2 V \, dx)^{\frac{1}{2}}$, and

$$\tilde{H}^1(M) = \{u \in W_{1,2}^{loc}(\Omega) : |u|_{1,2} := \left(|u|_2^2 + \|\nabla u\|_{L_2(\Omega)}^2\right)^{1/2} < \infty\}.$$

The closure of $C_0^1(\Omega)$ under this norm will be denoted by $\tilde{H}_0^1(M)$.

Let $\tilde{P}$ be the Friedrichs extension of the operator P considered as a symmetric operator in $\tilde{L}_2(M)$ with domain $C_0^1(\Omega)$ (see [1]).

Remark 3.1. If M is a complete Riemannian manifold, then the operator $\tilde{P}$ is the unique selfadjoint realization of P in $\tilde{L}_2(M)$. In this case, $\tilde{P}$ coincides with the Dirichlet realization of P with domain of definition given by

$$D(\tilde{P}) = \{u \mid u \in \tilde{L}_2(M) \cap H_{loc}^1(M), Pu \in \tilde{L}_2(M)\}.$$

We denote by $\sigma(\tilde{P})$, $\sigma_{point}(\tilde{P})$, the spectrum and point spectrum of $\tilde{P}$, respectively.

It is well known that

$$\lambda_0 := \inf \sigma(\tilde{P}) = \mu_\Omega = \inf_{u \in \tilde{H}_0^1(M)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 V \, dx},$$

and

$$\begin{aligned}
\lambda_0 &= \sup\{\lambda \in \mathbb{R} : \mathcal{C}_{P-\lambda}(\Omega) \neq \emptyset\} \\
&= \sup\{\lambda \in \mathbb{R} : \exists u \in H_{loc}^1(\Omega), u > 0, (P - \lambda)u \geq 0 \text{ in } \Omega\},
\end{aligned}$$

and the supremum λ_0 is achieved.

If the infimum in (1.1) is achieved, then it possesses a positive minimizer. Since every minimizer is a solution of the equation $(P - \lambda_0)u = 0$ in Ω, it follows that problem (1.1) possesses a minimizer φ if and only if $\lambda_0 = \mu_\Omega \in \sigma_{point}(\tilde{P})$ and $\varphi \in \mathcal{C}_{P-\lambda_0}(\Omega) \cap \tilde{L}_2(\Omega)$.

Proof of Lemma 1.1. By the Birman-Schwinger principle, $\lambda_0 \in \sigma_{point}(\tilde{P})$ if and only if there exists $\varphi \in \tilde{L}_2(M)$ such that for every $0 \leq \lambda < \lambda_0$ we have in the L_2 sense

$$(\lambda_0 - \lambda) \int_\Omega V(x)^{1/2} G_{-\Delta-\lambda V}^\Omega(x,y) V(y)^{1/2} \left(V(y)^{1/2}\varphi(y)\right) dy = V(x)^{1/2}\varphi(x). \tag{3.1}$$

Moreover, by the continuity of the minimizer φ and the positivity of V, (3.1) holds true if and only if

$$\int_\Omega G^\Omega_{-\Delta-\lambda V}(x,y)V(y)\varphi(y)\,dy = \frac{\varphi(x)}{\lambda_0-\lambda} \tag{3.2}$$

for all $x \in \Omega$.

Fix $x_0 \in \Omega$. Assume that $-\Delta-\lambda_0 V$ is subcritical in Ω. Since $\varphi \in \mathcal{C}_{P-\lambda_0}(\Omega)$ and $G^\Omega_{-\Delta-\lambda_0 V}(x_0,x)$ is a positive solution of the operator $-\Delta-\lambda_0 V$ of minimal growth in a neighborhood of infinity in Ω, it follows that there exists $C_\varepsilon > 0$ such that $G^\Omega_{-\Delta-\lambda_0 V}(x_0,x) \leq C_\varepsilon \varphi(x)$ for all $x \in \Omega \setminus B(x_0,\varepsilon)$. In particular,

$$G^\Omega_{-\Delta-\lambda_0 V}(x_0,y)V(y)\varphi(y) \in L_1(\Omega).$$

By the Lebesgue monotone convergence theorem

$$\int_\Omega G^\Omega_{-\Delta-\lambda_0 V}(x_0,y)V(y)\varphi(y)\,dy =$$

$$\lim_{\lambda\nearrow\lambda_0}\int_\Omega G^\Omega_{-\Delta-\lambda V}(x_0,y)V(y)\varphi(y)\,dy = \lim_{\lambda\nearrow\lambda_0}\frac{\varphi(x_0)}{\lambda_0-\lambda} = \infty,$$

which is a contradiction. Therefore $-\Delta-\lambda_0 V$ is critical, and φ is a ground state of the operator $-\Delta-\lambda_0 V$ in Ω. $\square$

4 Nonexistence of minimizers under small compact domain perturbations

In this section we prove Theorem 1.6 and give a direct proof of Corollary 1.7.

Proof. (**proof of Theorem 1.6**) Consider the domain $\Omega \subsetneq X$, and let $B_j \Subset X$ be the given decreasing sequence. Consider the Lipschitz portion $\Gamma \subset \partial\Omega$ such that $B_1 \cap \partial\Omega$ is contained in Γ.

Let Γ_ε denote the set

$$\Gamma_\varepsilon := \{x \in \Omega \mid \text{dist}\,(x,\Gamma) = \varepsilon\},$$

where $\varepsilon > 0$ is sufficiently small. Finally, fix $x_0 \in \Omega$ such that $\text{dist}\,(x_0,\Gamma) = \varepsilon/2$.

Suppose that that $\mu_j := \mu_{\Omega_j}$ is attained for all $j \geq 1$, and let $u_j \in \mathcal{C}_{\mu_j}(\Omega_j) \cap \mathcal{D}^{1,2}(\Omega_j)$ be the corresponding minimizer such that $u_j(x_0) = 1$. By Lemma 1.1, u_j is the normalized ground state of the (critical) operator $P_j := -\Delta - \mu_j V$ in Ω_j.

Clearly, $\mu_j \leq \mu_\Omega$. Therefore, P_j is subcritical in Ω, and denote by $G^\Omega_{P_j}(x, x_0)$ the corresponding positive minimal Green function.

Due to the local Harnack inequality, the behavior of the Green function near the pole x_0, and [11, Lemma 6.3], it follows that there exists $C > 0$ such that

$$C^{-1} G^\Omega_{P_j}(x, x_0) \leq u_j(x) \leq C G^\Omega_{P_j}(x, x_0),$$

for all $x \in \Gamma_\varepsilon$, and $j \geq 1$. Since u_j and $G^\Omega_{P_j}(x, x_0)$ are positive solutions of minimal growth of the operator P_j in a neighborhood of infinity in $\Omega \setminus \Gamma$ (see [11, Lemma 5.2]), it follows that

$$C^{-1} G^\Omega_{P_j}(x, x_0) \leq u_j(x) \leq C G^\Omega_{P_j}(x, x_0), \tag{4.1}$$

for all $x \in \Omega \cap \{\mathrm{dist}\,(x, \Gamma) > \varepsilon\}$ and $j \geq 1$.

By taking a subsequence, we may assume that $\mu_j \to \mu_0$, and $\{u_j\}$ converges in the open compact topology to a solution $u \in \mathcal{C}_{\mu_0}(\Omega)$. Clearly, $\mu_0 \leq \mu_\Omega$.

Since ∂B_j are smooth, and u_j vanish on $\partial B_j \cap \partial \Omega_j$, it follows by [5] and elliptic regularity that u vanishes on Γ.

Denote $P_0 := -\Delta - \mu_0 V$, and note that P_0 is subcritical in Ω. By (4.1) and the boundary Harnack principle,

$$C_1^{-1} G^\Omega_{P_0}(x, x_0) \leq u(x) \leq C_1 G^\Omega_{P_0}(x, x_0),$$

for all $x \in \Omega \setminus \{\mathrm{dist}\,(x, x_o) < \varepsilon/2\}$. Consequently, u is a global positive solution of the equation $P_0 u = 0$ in Ω which has minimal growth in a neighborhood of infinity in Ω. In other words, u is a ground state of the operator $-\Delta - \mu_0 V$ in Ω. But this is a contradiction, since for $\mu \leq \mu_\Omega$, the operator $-\Delta - \mu V$ is subcritical in Ω. $\qquad\qquad\square$

We conclude this section with a direct proof of Corollary 1.7.

Proof. (**proof of Corollary 1.7**) Let $C \subsetneq X$ be a Lipschitz cone, and let $D \subset S_1^{N-1}$ be the Lipschitz domain so that $C = \{(r, \omega) \,|\, r \in (0, \infty), \omega \in D\}$. Let $W \in L^p_{loc}(X)$ be a nonzero nonnegative function with a compact support in C, and set $V(x) = \frac{1}{|x|^2} - W(x)$. Clearly, $\mu_C = \frac{(N-2)^2}{4} + \lambda_D$, where λ_D

11

is the Dirichlet principal eigenvalue of $-\Delta_S$ on D. Moreover, the operator $-\Delta - \mu_C V$ is subcritical in C. Let $B_j \Subset X$ be a decreasing sequence of smooth domains, such that $C_j := B_j \cup C$ are connected for all $j \geq 1$, and $\mathrm{int}\,(\cap_j C_j) = C$. Fix $x_0 \in C$.

Suppose that that $\mu_j := \mu_{C_j}$ is attained for all $j \geq 1$, and let $u_j \in \mathcal{C}^0_{\mu_j}(C_j) \cap \mathcal{D}^{1,2}(C_j)$ be the corresponding minimizer such that $u_j(x_0) = 1$. By Lemma 1.1, u_j is a positive solution of the operator $-\Delta - \mu_j V$ of minimal growth in a neighborhood of infinity in C_j .

We denote

$$\alpha_{j,\pm} := \frac{-(N-2) \pm \sqrt{(N-2)^2 - 4(\mu_j - \lambda_D)}}{2}, \qquad v_{j,\pm}(x) := r^{\alpha_{j,\pm}} v_D(\omega).$$

where v_D is the Dirichlet principal eigenfunction of $-\Delta_S$ on D. Fix $0 < R_1 < R_2$ such that $\mathrm{supp}\,W \subset \{R_1 < |x| < R_2\}$ and

$$C_j \cap (\{|x| < R_1\} \cup \{|x| > R_2\}) \subset C.$$

By [11, Theorem 6.3], there exists $C > 0$ such that

$$C^{-1} v_{j,\pm}(x) \leq u_j(x) \leq C v_{j,\pm}(x),$$

for all $x \in C \cap (\{|x| = R_1\} \cup \{|x| = R_2\})$, and $j \geq 1$.

Since $u_j \in \mathcal{D}^{1,2}(C_j)$, and $v_{j,+}$ is a positive solution of minimal growth at the singular point $\zeta = 0$ of the operator $-\Delta - \frac{\mu_j}{|x|^2}$ in C, it follows that u_j is a positive solution of minimal growth at the singular point $\zeta = 0$, and

$$C^{-1} v_{j,+}(x) \leq u_j(x) \leq C v_{j,+}(x), \tag{4.2}$$

for all $x \in C \cap \{|x| < R_1\}\}$ and $j \geq 1$.

Similarly, since $u_j \in \mathcal{D}^{1,2}(C_j)$, and $v_{j,-}$ is a positive solution of minimal growth at the singular point $\zeta = \infty$ of the operator $-\Delta - \frac{\mu_j}{|x|^2}$ in C, it follows that u_j is a positive solution of minimal growth at the singular point $\zeta = \infty$, and

$$C^{-1} v_{j,-}(x) \leq u_j(x) \leq C v_{j,-}(x), \tag{4.3}$$

for all $x \in C \cap \{|x| > R_2\}\}$, and $j \geq 1$.

By taking a subsequence, we may assume that $\mu_j \to \mu_0$, and $\{u_j\}$ converges in the open compact topology to a solution $u \in \mathcal{C}_{\mu_0}(C)$. Moreover, since B_j are smooth, and u_j vanish on $\partial B_j \cap \partial C_j$, it follows by [5] and elliptic regularity that u vanishes on $\partial C \setminus \{0\}$. So, $u \in \mathcal{C}^0_{\mu_0}(C)$.

12

Clearly, $\mu_0 \leq \mu_C$. Furthermore, $\alpha_{j,\pm} \to \alpha_{0,\pm} := \frac{-(N-2)\pm\sqrt{(N-2)^2-4(\mu_0-\lambda_D)}}{2}$.
Therefore,
$$C^{-1}|x|^{\alpha_{0,+}}v_D(\frac{x}{|x|}) \leq u(x) \leq C|x|^{\alpha_{0,+}}v_D(\frac{x}{|x|}),$$
for all $x \in C \cap \{|x| < R_1\}\}$, and
$$C^{-1}|x|^{\alpha_{0,-}}v_D(\frac{x}{|x|}) \leq u(x) \leq C|x|^{\alpha_{0,-}}v_D(\frac{x}{|x|}),$$

for all $x \in C \cap \{|x| > R_2\}\}$. Consequently, u is a global positive solution of the equation $(-\Delta - \mu_0 V)u = 0$ in C which has minimal growth in a neighborhood of infinity in C. In other words, u is a ground state of the operator $-\Delta - \mu_0 V$ in C. But this is a contradiction, since for $\mu \leq \mu_C$, the operator $-\Delta - \mu V$ is subcritical in C. $\square$

Acknowledgments

References

[1] S. Agmon, Bounds on exponential decay of eigenfunctions, *in* "Schrödinger Operators", ed. S. Graffi, Lecture Notes in Math. Vol. 1159, Springer-Verlag, Berlin, 1985, pp. 1–38.

[2] H. Brezis and M. Marcus, Hardy's inequalities revisited, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **25** (1997), 217–237.

[3] H. Brezis, M. Marcus and I. Shafrir, Extremal functions for Hardy's inequality with weight, *J. Funct. Anal.* **171** (2000), 177–191.

[4] F. Colin and Y. Hupperts, Minimization problems related to generalized Hardy's inequalities, *Nonlinear Anal.* **52** (2003), 1933–1945.

[5] D. Daners, Dirichlet problems on varying domains, *J. Differential Equations* **188** (2003), 591–624.

[6] M. Marcus, V. J. Mizel, and Y. Pinchover, On the best constant for Hardy's inequality in $\mathbb{R}^n$, *Trans. Amer. Math. Soc.* **350** (1998), 3237–3255.

[7] M. Marcus and I. Shafrir, An eigenvalue problem related to Hardy's L^p inequality. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **29** (2000), 581–604.

[8] U. Mosco, Convergence of convex sets and of solutions of variational inequalities, *Advances in Math.* **3** (1969), 510–585.

[9] M. Murata, Structure of positive solutions to $(-\Delta + V)u = 0$ in $\mathbb{R}^n$, *Duke Math. J.* **53** (1986), 869–943.

[10] Y. Pinchover, Large time behavior of the heat kernel and the behavior of the Green function near criticality for nonsymmetric elliptic operators, *J. Funct. Anal.* **104** (1992), 54–70.

[11] Y. Pinchover, On positive Liouville theorems and asymptotic behavior of solutions of Fuchsian type elliptic operators, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **11** (1994), 313–341.

[12] R. G. Pinsky, "Positive Harmonic Functions and Diffusion", Cambridge Studies in Advanced Mathematics, Vol. 45, Cambridge University Press, Cambridge, 1995.

[13] D. Smets, A concentration-compactness lemma with applications to singular eigenvalue problems, *J. Func. Anal.* **167** (1999), 463–480.

[14] A. Tertikas, Critical phenomena in linear elliptic problems, *J. Func. Anal.* **154** (1998), 42–66.